Infinite series

https://www.linkedin.com/groups/8313943/8313943-6427099519065948163

Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that the infinite series $\sum_{n=1}^{\infty} \frac{H_{n+1}}{n(n+1)}$ converges and find its value.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Note that $\sum_{k=1}^{n} \frac{H_{k+1}}{k(k+1)} = \sum_{k=1}^{n} H_{k+1}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{n} \frac{H_{k+1}}{k} - \sum_{n=1}^{\infty} \frac{H_{k+1}}{k+1} =$

$\sum_{k=1}^{n} \frac{H_{k+1}}{k} - \sum_{k=2}^{n+1} \frac{H_k}{k} = \frac{H_2}{1} + \sum_{k=2}^{n} \frac{H_{k+1}}{k} - \sum_{k=2}^{n} \frac{H_k}{k} - \frac{H_{n+1}}{n+1} =$

$1 + \frac{1}{2} + \sum_{k=2}^{n} \frac{1}{k}(H_{k+1} - H_k) - \frac{H_{n+1}}{n+1} = \frac{3}{2} + \sum_{k=2}^{n} \frac{1}{k(k+1)} - \frac{H_{n+1}}{n+1}.$

Since $\sum_{k=2}^{n} \frac{1}{k(k+1)} = \sum_{k=2}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2} - \frac{1}{n+1}$ then

$\sum_{k=1}^{n} \frac{H_{k+1}}{k(k+1)} = \frac{3}{2} + \frac{1}{2} - \frac{1}{n+1} - \frac{H_{n+1}}{n+1} = 2 - \frac{H_{n+1}+1}{n+1}.$

We will prove that $\lim_{n\to\infty} \frac{H_n}{n} = 0.$

By AM-QM Inequality $\frac{H_n}{n} \le \sqrt{\frac{1}{n}\sum_{k=1}^{n} \frac{1}{k^2}}$ and since $\sum_{k=1}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \frac{1}{(k-1)k} =$

$1 + \sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 + 1 - \frac{1}{n} < 2$ then $\frac{H_n}{n} < \sqrt{\frac{2}{n}}$ and, therefore,

$\lim_{n\to\infty} \frac{H_n}{n} = 0.$ Hence, $\sum_{n=1}^{\infty} \frac{H_{n+1}}{n(n+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{H_{k+1}}{k(k+1)} = \lim_{n\to\infty}\left(2 - \frac{H_{n+1}+1}{n+1}\right) = 2.$